# Towards a test particle description of transport processes for states with continuous mass spectra

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## Abstract

Based on a first order gradient expansion a consistent transport equation is derived for a nonrelativistic system beyond the quasiparticle approximation, i.e. for a regime where the dynamically generated width of the states is allowed to be large. An exactly conserved quantity is identified which is interpreted as an effective particle number obtained by coarse graining. Using a test particle ansatz for this conserved quantity allows to rewrite the transport equation into equations of motion for test particles. The two-body collision terms are formulated in terms of the test particles which gain non-trivial renormalization factors due to the coarse graining process.

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## I. INTRODUCTION

Since the pioneering works on non-equilibrium quantum field theory [1–4] semiclassical transport theory has become a major tool to solve various problems of many-body physics. While former works have focused their attention more or less on the quasi-particle regime (see e.g. [5–11] and references therein) and small corrections to it [12–14] the extension of the formalism to off-shell phenomena has become a topic of growing interest in the last few years. E.g. in heavy-ion collisions it has turned out that the collision rates usually are so high that an on-shell approximation seems to be inappropriate [15–18]. In addition, the resonances excited during the reaction may have large decay width. Therefore, a representation of these states by stable particles may not be a proper approximation.

In this article we shall set up a formalism which allows for a consistent transport theoretical description of states with a continuous mass spectrum. For simplicity we restrict ourselves to the non-relativistic case. We will comment in the last section on possible generalizations to relativistic systems. We will derive a transport equation in first order gradient expansion which includes off-shell effects. We will discuss in detail how this transport equation can be solved by a proper test particle ansatz. To account properly for the off-shell behavior the test particles are allowed to have an arbitrary energy not restricted by the mass-shell condition. While a test particle representation of a transport equation in the quasi-particle limit is straightforward we will see that this becomes a rather non-trivial issue if off-shell effects are included. Especially we have to worry about a quantity which is fully conserved by the transport equation. Only such a quantity can be represented by test particles. Otherwise the test particle ansatz would yield a number of equations which is higher than the number of test particle coordinates, i.e. an overdetermined system of equations. We will isolate a quantity which is fully conserved by the transport equation and derive the corresponding test particle equations of motion. Finally we will comment on the form of the collision terms for these test particles.

The article is organized as follows. In the next section we discuss in some detail for a simple example how a test particle ansatz works for a conserved quantity and fails for a non-conserved one. In Sec. III equations of motion for the quantities of interest are derived from the underlying quantum field theoretical Kadanoff-Baym equations by performing a first order gradient expansion. This derivation is very similar to the one presented recently in [16] (second reference). Therefore, Sec. III might be seen as a review. It is included to keep the article self-contained and to put emphasis on various other aspects as compared to previous works. In Sec. IV new material is presented. Especially a quantity is isolated which is conserved by the transport equation derived in the preceeding section. For this quantity a test particle ansatz is presented in Sec. V and the equations of motion for the test particles are derived. Finally we evaluate the self-energies in the Born approximation using the test particle representation. We will summarize our results in Sec. VI, compare them to previous works and give an outlook on unresolved problems.

## II. TEST PARTICLE ANSATZ

Before turning to the proper transport equation for modes with a continuous mass spectrum we would like to discuss how a test particle ansatz for the quantity of interest works and for which cases it makes sense. For this aim let us consider an equation for a quantity $f(t,p)$ which is too complicated to solve it exactly. The only thing we have to know about this exact equation is that it conserves a particle number

$$N(t) = \int dp\, f(t,p)\,. \tag{2.1}$$

By an approximation scheme one obtains from the exact equation the following approximate one (which resembles a Vlasov equation):

$$\frac{\partial}{\partial t}\left[(1-\kappa(t,p))f(t,p)\right] + F(t)\,\frac{\partial}{\partial p} f(t,p) = 0 \tag{2.2}$$

with a force term $F(t)$ and a renormalization $\kappa$ which both are not further specified. Obviously this approximate (transport) equation does *not* conserve the particle number $N$ but instead the quantity

$$\tilde{N}(t) = \int dp\,(1-\kappa(t,p))f(t,p)\,. \tag{2.3}$$

We now discuss two cases to solve (2.2), namely test particle ansätze for $f$ and for $\tilde{f} = (1-\kappa)f$, respectively. In general, a test particle representation for any of the two quantities is given by

$$\left.\begin{array}{c} f \\ \text{or} \\ \tilde{f} \end{array}\right\} = \frac{1}{L}\sum_{i=1}^{M}\delta(p-p_i(t)) \tag{2.4}$$

where $L$ is a normalization constant, $i$ numbers the $M$ test particles and $p_i(t)$ characterizes the trajectory of the test particle $i$. Obviously the momentum integral over (2.4) is conserved and given by $M/L$. Therefore, a test particle ansatz for $f$ to solve (2.2) seems to be inappropriate since the momentum integral over $f$ (which yields the particle number) is not conserved by (2.2). Nonetheless, it is instructive to figure out which equations one gets with a test particle ansatz for $f$. In this case one finds

$$0 = \sum_i \left\{ -\frac{d\kappa(t,p_i(t))}{dt}\,\delta(p-p_i(t)) + \left[F(t) - [1-\kappa(t,p_i(t))]\dot{p}_i(t)\right]\frac{\partial}{\partial p}\delta(p-p_i(t))\right\}. \tag{2.5}$$

Note the appearance of the total derivative with respect to $t$ in contrast to the partial derivative in (2.2). This is obtained by

$$\begin{aligned}
&\frac{\partial}{\partial t}[(1-\kappa(t,p))\,\delta(p-p_i)] = -\frac{\partial\kappa(t,p)}{\partial t}\,\delta(p-p_i) + (1-\kappa(t,p))\,\frac{\partial}{\partial t}\delta(p-p_i)\\
&= -\frac{\partial\kappa(t,p)}{\partial t}\,\delta(p-p_i) + (1-\kappa(t,p))\,\frac{\partial}{\partial p}\delta(p-p_i)\,(-\dot{p}_i)\\
&= -\frac{\partial\kappa(t,p_i)}{\partial t}\,\delta(p-p_i) - \dot{p}_i\,\frac{\partial\kappa(t,p_i)}{\partial p_i}\,\delta(p-p_i) - (1-\kappa(t,p_i))\,\dot{p}_i\,\frac{\partial}{\partial p}\delta(p-p_i)\\
&= -\frac{d\kappa(t,p_i)}{dt}\,\delta(p-p_i) - (1-\kappa(t,p_i))\,\dot{p}_i\,\frac{\partial}{\partial p}\delta(p-p_i)
\end{aligned} \tag{2.6}$$

where we have used the identity

$$h(x)\,\frac{\partial}{\partial x}\delta(x-y) = h(y)\,\frac{\partial}{\partial x}\delta(x-y) - \frac{dh(y)}{dy}\,\delta(x-y)\,. \tag{2.7}$$

To fulfill (2.5) one has to demand that the coefficient of the $\delta$-function as well as the coefficient of the momentum derivative of the $\delta$-function both vanish, i.e.

$$-\frac{d\kappa(t,p_i(t))}{dt} = 0\,, \tag{2.8}$$

$$F(t) - [1-\kappa(t,p_i(t))]\dot{p}_i(t) = 0\,. \tag{2.9}$$

In this way we obtain *two* equations of motion for *one* test particle coordinate:

$$\dot{p}_i(t) = -\frac{\dfrac{\partial\kappa(t,p_i)}{\partial t}}{\dfrac{\partial\kappa(t,p_i)}{\partial p_i}}\,, \tag{2.10}$$

$$\dot{p}_i(t) = \frac{F(t)}{1-\kappa(t,p_i)}\,. \tag{2.11}$$

Obviously, this provides an overdetermined system which is simply due to the fact that a test particle ansatz has been made for a quantity which is not conserved by the equation one wants to solve. On the other hand, a test particle ansatz for $\tilde{f}$ yields

$$0 = \sum_i \left[\frac{F(t)}{1-\kappa(t,p_i(t))} - \dot{p}_i(t)\right]\frac{\partial}{\partial p}\delta(p-p_i(t)) \tag{2.12}$$

Here the coefficient of the pure $\delta$-function vanishes. This is a general property if a test particle ansatz for a conserved quantity is made. We only have to fulfill one equation:

$$\dot{p}_i(t) = \frac{F(t)}{1-\kappa(t,p_i)}\,. \tag{2.13}$$

Here we observe a second interesting feature of a test particle ansatz. The last equation is identical to the second one of the equations obtained before. Both equations (2.11) and (2.13) were obtained by demanding that the respective coefficient of the $\partial_p\delta(p-p_i)$ term has to vanish. Thus, the difference between a test particle ansatz for a conserved quantity and for a non-conserved quantity lies only in the appearance of a term with a pure $\delta$-function (i.e. without derivatives acting on it). This provides an additional equation which causes the system of equations to be overdetermined for the case of a non-conserved quantity. The other equations do not differ for both cases.

So far we were only concerned with a proper solution of the transport equation (2.2) and indeed we have presented one consistent way to solve it, namely by a test particle ansatz for $\tilde{f}$. Let us now recall that (2.2) is only an approximate equation and that the exact equation in contrast to the transport equation exactly conserves $f$. Thus, one has to realize that the two equations (2.10,2.11) are identical to each other up to terms which have been neglected when the transport equation (2.2) was derived from the underlying exact equation.

Therefore, one might conclude that one can use one or the other equation to evolve the test particle coordinates and thus $f$ in time. The difference between these two evolutions is in an order which was neglected anyway when deriving the transport equation from the exact one. Therefore, we now choose e.g. (2.11) to evolve $f$. One has to realize, however, that now it is no longer the transport equation (2.2) one is solving. Instead, one can reconstruct from the test particle equation of motion that the corresponding transport equation is given by

$$\begin{aligned} 0 &= \sum_i \left[ \frac{F(t)}{1-\kappa(t,p_i(t))} - \dot{p}_i(t) \right] \frac{\partial}{\partial p} \delta(p-p_i(t)) \\ &= \frac{\partial}{\partial p} \left[ \frac{F(t)}{1-\kappa(t,p)} \sum_i \delta(p-p_i(t)) \right] + \frac{\partial}{\partial t} \sum_i \delta(p-p_i(t)) \\ &= F(t) \frac{\partial}{\partial p} \frac{f(t,p)}{1-\kappa(t,p)} + \frac{\partial}{\partial t} f(t,p) \end{aligned} \tag{2.14}$$

which clearly differs from (2.2). However, by construction this new transport equation (2.14) is as close to the underlying exact equation as the original transport equation (2.2).

To summarize we have presented two ways to deal with the approximate transport equation and the fact that the particle number is conserved by the exact equation.

1. Extract a quantity which is exactly conserved by the approximate equation (here $\tilde{N}$). Make a test particle ansatz for the corresponding density (here $\tilde{f}$). Once the transport equation is solved one can also calculate $f$ and $N$ and figure out to which extent the conservation of $N$ is violated in time. This provides a check for the accuracy of the approximation scheme which has served to derive the transport equation from the exact one.

2. Make a test particle ansatz for the quantity which is exactly conserved by the exact equation but not by the transport equation (here $f$). Ignore one of the obtained equations of motion for the test particle coordinates. Usually one is tempted to ignore the equation obtained from the coefficient of the pure $\delta$-function. Reconstruct a new transport equation from the test particle equations of motion.

At first sight the second approach seems to be more appealing since there the reconstructed transport equation shares with the exact equation the full conservation of the particle number. Unfortunately things are not always as simple as the chosen example seems to indicate. The problem is that in general it is not always possible to reconstruct such a transport equation from the test particle equations. As long as only the propagation of test particles is concerned (Vlasov-type equation) one can reconstruct a corresponding transport equation as shown above. If, however, also collisions of test particles are taken into account (Boltzmann-type equations) new problems arise. To see this we extend (2.2) by adding a collision term on the r.h.s.

$$\frac{\partial}{\partial t} \left[ (1-\kappa(t,p)) f(t,p) \right] + F(t) \frac{\partial}{\partial p} f(t,p) = I_{\text{coll}}[f] \tag{2.15}$$

with the additional condition

$$\int dp\, I_{\rm coll}[f] = 0\,. \tag{2.16}$$

We stress again that this equation has to be regarded as an approximate (transport) equation. For the underlying exact equation we assume that it is too complicated to solve and that it conserves the particle number $N$. Again we observe that $\tilde{N}$ is conserved by (2.15) while $N$ is not conserved. The most pragmatic (albeit doubtful) approach would be the following: One makes a test particle ansatz for $f$. For the propagation of the test particles *between collisions* one obtains again the overdetermined system of equations (2.10,2.11). One disregards the first equation and uses the second one to evolve the test particle coordinates in time. One allows in addition for collisions between the particles according to the collision term on the r.h.s. of (2.15). The obstacle in that pragmatic approach is the fact that there is *no* corresponding transport equation at all which can be derived from (2.15). Reconstructing a transport equation from the test particle equation of motion yields

$$\frac{\partial}{\partial t} f(t,p) + F(t)\,\frac{\partial}{\partial p}\frac{f(t,p)}{1-\kappa(t,p)} = \frac{I_{\rm coll}[f]}{1-\kappa(t,p)}\,. \tag{2.17}$$

Thus the collision term now is given by $I_{\rm coll}/(1-\kappa)$ instead of $I_{\rm coll}$. This new collision term, however, does not conserve the particle number

$$\frac{dN}{dt} = \frac{d}{dt}\int dp\, f(t,p) = \int dp\, \frac{I_{\rm coll}[f]}{1-\kappa(t,p)} \neq 0\,. \tag{2.18}$$

Therefore, the reconstructed transport equation (2.17) has no advantages as compared to the original one (2.15). Using instead the original collision term $I_{\rm coll}$ as described above is only a guess. It is not clear whether such an ad hoc postulated equation is as close to the exact equation as the transport equation (2.15).

Aiming at a field theoretical foundation for the description of transport processes we think it is not acceptable to have only a recipe for the test particle evolution which is not derivable from the underlying theory. If an already approximate transport equation is only approximately solved one loses the contact to the original exact equation more and more in a not controllable way. Therefore, we suggest to use the first approach described above, i.e. the test particle ansatz for $\tilde{f}$, since it solves the transport equation exactly. In addition, it provides the possibility to judge the approximation scheme which has led to the transport equation simply by evaluating the time dependence of the quantity which is conserved by the exact equation.

After these precursory considerations concerning the use of the test particle ansatz we now turn to the derivation of a transport equation from the underlying field-theoretical Kadanoff-Baym equations.

## III. KADANOFF-BAYM EQUATIONS AND GRADIENT EXPANSION

Following Kadanoff and Baym [2] we start with the exact nonrelativistic equations of motion for the two-point functions

$$\left(i\frac{\partial}{\partial t_1}+\frac{\Delta_1}{2m}\right)D^<(1,1')=\int d\bar{1}\left[\Sigma^{\rm ret}(1,\bar{1})\,D^<(\bar{1},1')+\Sigma^<(1,\bar{1})\,D^{\rm av}(\bar{1},1')\right]\,, \tag{3.1}$$

$$\left(i\frac{\partial}{\partial t_1}+\frac{\Delta_1}{2m}\right)D^>(1,1')=\int d\bar{1}\left[\Sigma^{\rm ret}(1,\bar{1})\,D^>(\bar{1},1')+\Sigma^>(1,\bar{1})\,D^{\rm av}(\bar{1},1')\right] \tag{3.2}$$

where we have introduced the two-point functions without ordering

$$iD^<(x,y)=\pm\langle\psi^\dagger(y)\,\psi(x)\rangle\,, \tag{3.3}$$

$$iD^>(x,y)=\quad\langle\psi(x)\,\psi^\dagger(y)\rangle \tag{3.4}$$

and the retarded and advanced quantities

$$D^{\rm ret}(x,y)=\Theta(x_0-y_0)\,[D^>(x,y)-D^<(x,y)]\,, \tag{3.5}$$

$$D^{\rm av}(x,y)=\Theta(y_0-x_0)\,[D^<(x,y)-D^>(x,y)]\,. \tag{3.6}$$

In (3.3) and throughout this work the upper (lower) sign refers to bosons (fermions) except where otherwise stated. The self-energies are connected via

$$\Sigma^{\rm ret}(x,y)=\Sigma^{\rm HF}(x,y)+\Theta(x_0-y_0)\,[\Sigma^>(x,y)-\Sigma^<(x,y)]\,, \tag{3.7}$$

$$\Sigma^{\rm av}(x,y)=\Sigma^{\rm HF}(x,y)+\Theta(y_0-x_0)\,[\Sigma^<(x,y)-\Sigma^>(x,y)]\,. \tag{3.8}$$

The collisional self-energies $\Sigma^<$ and $\Sigma^>$ will be specified below. We have also introduced the time-local Hartree-Fock self-energy $\Sigma^{\rm HF}(x,y)\sim\delta(x_0-y_0)$. In (3.1,3.2) the numbers denote short-hand notations for the space-time coordinates.

In the following we will concentrate on systems where the dependence of an arbitrary two-point function or self-energy $F(x,y)$ on its center-of-mass variable $(x+y)/2$ is weak (albeit not negligible). Therefore it is appropriate to introduce the Fourier transform with respect to the more rapidly oscillating difference variable $x-y$ (Wigner transformation):

$$\bar{F}(X,p)=\int d^4u\,e^{ipu}F(X+u/2,X-u/2) \tag{3.9}$$

where $F$ denotes an arbitrary two-point function or self-energy,

$$F=D^<,\,D^>,\ldots,\,\Sigma^<,\ldots\,. \tag{3.10}$$

From the definitions (3.3,3.4,3.5,3.6,3.9) we obtain the following transformation properties with respect to complex conjugation:

$$\begin{aligned}\left[\bar{D}^<(X,p)\right]^*&=-\bar{D}^<(X,p)\,,\\ \left[\bar{D}^>(X,p)\right]^*&=-\bar{D}^>(X,p)\,,\\ \left[\bar{D}^{\rm ret}(X,p)\right]^*&=\bar{D}^{\rm av}(X,p)\,.\end{aligned} \tag{3.11}$$

Corresponding equations hold also for the self-energies. It is useful to introduce the real-valued quantities

$$S^<(X,p)=\pm i\bar{D}^<(X,p)\,,\qquad S^>(X,p)=i\bar{D}^>(X,p) \tag{3.12}$$

and

$$\begin{aligned}\mathcal{A}(X,p) &= -2\mathrm{Im}\bar{D}^{\mathrm{ret}}(X,p) = 2\mathrm{Im}\bar{D}^{\mathrm{av}}(X,p) \\ &= i[\bar{D}^{\mathrm{ret}}(X,p) - \bar{D}^{\mathrm{av}}(X,p)] = i[\bar{D}^{>}(X,p) - \bar{D}^{<}(X,p)] \\ &= S^{>}(X,p) \mp S^{<}(X,p)\,. \end{aligned} \tag{3.13}$$

The quantity $\mathcal{A}$ is the spectral function. On account of the (anti-)commutation relation for bosonic (fermionic) fields

$$\psi(t,\vec{x})\,\psi^{\dagger}(t,\vec{y}) \mp \psi^{\dagger}(t,\vec{y})\,\psi(t,\vec{x}) = \delta(\vec{x}-\vec{y}) \tag{3.14}$$

the spectral function is normalized to one:

$$\int\frac{dp_0}{2\pi}\mathcal{A}(X,p) = 1\,. \tag{3.15}$$

The quantity $S^{<}$ can be interpreted as a generalized phase-space density. The particle number can be obtained from $S^{<}$ as

$$N(t) = \int d^3x \int\frac{d^4p}{(2\pi)^4} S^{<}(t,\vec{x};p)\,. \tag{3.16}$$

For a system which has reached thermal equilibrium $S^{<}(X,p)$ does no longer depend on $X$ and is given by

$$S^{<}_{\mathrm{th}}(p) = n_{\mathrm{B,F}}(p_0)\,\mathcal{A}(p) \tag{3.17}$$

where $n_{\mathrm{B,F}}$ is the thermal Bose (Fermi) distribution.

For a general off-equilibrium situation we are aiming at equations of motion for the real-valued quantities $S^{<}$, $\mathcal{A}$, and $\mathrm{Re}\bar{D}^{\mathrm{ret}} = \mathrm{Re}\bar{D}^{\mathrm{av}}$. (Note that $S^{>}$ can be obtained from $S^{<}$ and $\mathcal{A}$ according to (3.13)). In principle, $\mathrm{Re}\bar{D}^{\mathrm{ret}}$ can be obtained from $\mathcal{A}$ via a dispersion relation which can be easily obtained by Wigner transformation of the definitions (3.5,3.6):

$$\mathrm{Re}\bar{D}^{\mathrm{ret}}(X,p) = \int\frac{dk_0}{2\pi}\mathcal{P}\frac{1}{p_0-k_0}\mathcal{A}(X;k_0,\vec{p}) \tag{3.18}$$

where $\mathcal{P}$ denotes the principal value. It is easier, however, to derive the equation of motion for $\mathrm{Re}\bar{D}^{\mathrm{ret}}$ from an equation of motion for $D^{\mathrm{ret}}$. Using (3.1,3.2) and the definition (3.5) we get

$$\left(i\frac{\partial}{\partial t_1} + \frac{\Delta_1}{2m}\right) D^{\mathrm{ret}}(1,1') = \delta^{(4)}(1,1') + \int d\bar{1}\,\Sigma^{\mathrm{ret}}(1,\bar{1})\,D^{\mathrm{ret}}(\bar{1},1')\,. \tag{3.19}$$

As we will see below the final result for $\mathrm{Re}\bar{D}^{\mathrm{ret}}$ agrees with (3.18).

As already mentioned we are interested in systems where the dependence of the two-point functions on $X$ is weak. This allows to Wigner transform the equations of motion (3.1,3.2,3.19) and to neglect all terms with more than one derivative with respect to the center-of-mass variable $X$. We get the following equations of motion in first order gradient expansion:

$$\left(p_0 - \frac{\vec{p}^2}{2m}\right) \bar{D}^<(X,p) - \frac{i}{2}\,[p_0 - \frac{\vec{p}^2}{2m},\, \bar{D}^<(X,p)] =$$
$$\bar{\Sigma}^{\rm ret}(X,p)\,\bar{D}^<(X,p) + \bar{\Sigma}^<(X,p)\,\bar{D}^{\rm av}(X,p)$$
$$- \frac{i}{2}\,[\bar{\Sigma}^{\rm ret}(X,p),\, \bar{D}^<(X,p)] - \frac{i}{2}\,[\bar{\Sigma}^<(X,p),\, \bar{D}^{\rm av}(X,p)]\,, \tag{3.20}$$

$$\left(p_0 - \frac{\vec{p}^2}{2m}\right) \bar{D}^>(X,p) - \frac{i}{2}\,[p_0 - \frac{\vec{p}^2}{2m},\, \bar{D}^>(X,p)] =$$
$$\bar{\Sigma}^{\rm ret}(X,p)\,\bar{D}^>(X,p) + \bar{\Sigma}^>(X,p)\,\bar{D}^{\rm av}(X,p)$$
$$- \frac{i}{2}\,[\bar{\Sigma}^{\rm ret}(X,p),\, \bar{D}^>(X,p)] - \frac{i}{2}\,[\bar{\Sigma}^>(X,p),\, \bar{D}^{\rm av}(X,p)] \tag{3.21}$$

$$\left(p_0 - \frac{\vec{p}^2}{2m}\right) \bar{D}^{\rm ret}(X,p) - \frac{i}{2}\,[p_0 - \frac{\vec{p}^2}{2m},\, \bar{D}^{\rm ret}(X,p)] =$$
$$1 + \bar{\Sigma}^{\rm ret}(X,p)\,\bar{D}^{\rm ret}(X,p) - \frac{i}{2}\,[\bar{\Sigma}^{\rm ret}(X,p),\, \bar{D}^{\rm ret}(X,p)] \tag{3.22}$$

where we have introduced the generalized Poisson bracket

$$[A,B] = \partial_{X_0} A\,\partial_{p_0} B - \partial_{p_0} A \partial_{X_0} B - \vec{\nabla}_X A\,\vec{\nabla}_p B + \vec{\nabla}_p A\,\vec{\nabla}_X B\,. \tag{3.23}$$

It is worth noting that the drift terms typical for transport equations show up in the expressions

$$[p_0 - \frac{\vec{p}^2}{2m},\, B] = -\partial_{X_0} B - \frac{\vec{p}}{m}\cdot\vec{\nabla}_X B\,. \tag{3.24}$$

We note that for the derivation of (3.20-3.22) we have not assumed anything about the derivatives with respect to $p$. This is different to the approach of [7] where systems were studied for which mixed double derivatives like $\vec{\nabla}_X\vec{\nabla}_p A$ are also negligibly small. We will come back to that subtle difference below.

Next we extract from (3.20-3.22) real and imaginary parts using (3.12,3.13) and the decompositions

$$\bar{\Sigma}^{\rm ret/av}(X,p) = {\rm Re}\bar{\Sigma}^{\rm ret/av}(X,p) \mp \frac{1}{2}\,i\,\Gamma(X,p) \tag{3.25}$$

where we have introduced the (real-valued, non-negative) width

$$\Gamma = i(\bar{\Sigma}^> - \bar{\Sigma}^<) = i(\bar{\Sigma}^{\rm ret} - \bar{\Sigma}^{\rm av})\,. \tag{3.26}$$

In (3.25) the minus (plus) sign refers to the retarded (advanced) self-energy. We find

$$\left(p_0 - \frac{\vec{p}^2}{2m}\right) S^< = {\rm Re}\bar{\Sigma}^{\rm ret}\,S^< \pm i\bar{\Sigma}^<\,{\rm Re}\bar{D}^{\rm ret} - \frac{1}{4}[\Gamma,\, S^<] + \frac{1}{4}[\pm i\bar{\Sigma}^<,\, \mathcal{A}]\,, \tag{3.27}$$

$$[p_0 - \frac{\vec{p}^2}{2m},\, S^<] = \Gamma\,S^< \mp i\bar{\Sigma}^<\,\mathcal{A} + [{\rm Re}\bar{\Sigma}^{\rm ret},\, S^<] + [\pm i\bar{\Sigma}^<,\, {\rm Re}\bar{D}^{\rm ret}]\,, \tag{3.28}$$

$$\left(p_0 - \frac{\vec{p}^2}{2m}\right)\mathcal{A} = \mathrm{Re}\bar{\Sigma}^{\mathrm{ret}}\,\mathcal{A} + \Gamma\,\mathrm{Re}\bar{D}^{\mathrm{ret}}\,, \tag{3.29}$$

$$[p_0 - \frac{\vec{p}^2}{2m},\,\mathcal{A}] = [\mathrm{Re}\bar{\Sigma}^{\mathrm{ret}},\,\mathcal{A}] + [\Gamma,\,\mathrm{Re}\bar{D}^{\mathrm{ret}}]\,, \tag{3.30}$$

$$\left(p_0 - \frac{\vec{p}^2}{2m}\right)\mathrm{Re}\bar{D}^{\mathrm{ret}} = 1 + \mathrm{Re}\bar{\Sigma}^{\mathrm{ret}}\,\mathrm{Re}\bar{D}^{\mathrm{ret}} - \frac{1}{4}\Gamma\,\mathcal{A}\,, \tag{3.31}$$

$$[p_0 - \frac{\vec{p}^2}{2m},\,\mathrm{Re}\bar{D}^{\mathrm{ret}}] = [\mathrm{Re}\bar{\Sigma}^{\mathrm{ret}},\,\mathrm{Re}\bar{D}^{\mathrm{ret}}] - \frac{1}{4}[\Gamma,\,\mathcal{A}]\,. \tag{3.32}$$

Note that $i\bar{\Sigma}^<$ is real-valued (cf. Eq. (3.11) and the remark concerning self-energies).

Obviously using the gradient expansion up to first order we have obtained six equations for three quantities ($S^<$, $\mathcal{A}$, and $\mathrm{Re}\bar{D}^{\mathrm{ret}}$). Therefore some of these six equations are either redundant or provide constraints which have to be fulfilled by the dynamical quantities to make sure that their evolution is in accordance with the first order gradient expansion. To say it in other words: Even if one starts out with a configuration (represented by the two-point functions at initial time) for which the dependence on $X$ is weak, a peculiar choice for the self-energies might drive the system out of the regime where the gradient expansion is valid. Thus it would not be surprising if the gradient expanded equations of motion provided constraints on the dynamical quantities.

As we will see now there are indeed redundant equations as well as constraints. Let us start with the purely algebraic equations (3.29,3.31). They already yield the following expressions for two of the three quantities of interest:

$$\mathcal{A}(X,p) = \frac{\Gamma(X,p)}{\left(p_0 - \frac{\vec{p}^2}{2m} - \mathrm{Re}\bar{\Sigma}^{\mathrm{ret}}(X,p)\right)^2 + \frac{1}{4}\Gamma^2(X,p)}\,, \tag{3.33}$$

$$\mathrm{Re}\bar{D}^{\mathrm{ret}}(X,p) = \frac{p_0 - \frac{\vec{p}^2}{2m} - \mathrm{Re}\bar{\Sigma}^{\mathrm{ret}}(X,p)}{\left(p_0 - \frac{\vec{p}^2}{2m} - \mathrm{Re}\bar{\Sigma}^{\mathrm{ret}}(X,p)\right)^2 + \frac{1}{4}\Gamma^2(X,p)}\,. \tag{3.34}$$

For the retarded and advanced two-point functions this yields

$$\bar{D}^{\mathrm{ret/av}}(X,p) = \frac{1}{p_0 - \frac{\vec{p}^2}{2m} - \bar{\Sigma}^{\mathrm{ret/av}}(X,p)}\,. \tag{3.35}$$

These relations are very well known for systems which are completely homogeneous in space and time, i.e. where the zeroth order in the gradient expansion is sufficient. It is interesting to observe now that these relations are still valid in first order gradient expansion [15,11]. We also note that inserting (3.33) in (3.18) yields (3.34) provided that the used self-energy is indeed retarded, i.e. satisfies the dispersion relation

$$\mathrm{Re}\bar{\Sigma}^{\mathrm{ret}}(X;p_0,\vec{p}) = \bar{\Sigma}^{\mathrm{HF}}(X;\vec{p}) + \int\frac{dk_0}{2\pi}\mathcal{P}\frac{1}{p_0 - k_0}\Gamma(X;k_0,\vec{p})\,. \tag{3.36}$$

Finally one can show that the expression for the spectral function (3.33) indeed fulfills the normalization condition (3.15). This is discussed in Appendix A.

Next, we observe that the expressions (3.33,3.34) identically solve (3.30,3.32). To see this one might simply insert (3.33,3.34) and perform the somewhat tedious brute-force calculation. Instead we will take here the following elegant way [19]: We combine (3.30,3.32) to a complex equation. (3.32)$-\frac{i}{2}$(3.30) yields

$$[p_0 - \frac{\vec{p}^2}{2m} - \bar{\Sigma}^{\rm ret},\, \bar{D}^{\rm ret}] = 0\,. \tag{3.37}$$

It is trivial to see that the expression (3.35) solves this equation. It is interesting to note here that the same reasoning holds for relativistic scalar bosons. For relativistic fermions, however, things might be more complicated since in this case the generalized Poisson bracket of the matrix valued quantities $\bar{D}^{\rm ret}$ and $(\bar{D}^{\rm ret})^{-1}$ may not vanish any more [19]. To the best of our knowledge this subtlety has never been worked out in detail (concerning a closely related case cf. [20]).

We conclude that the equations (3.30,3.32) provide no additional information as compared to (3.29,3.31); they are redundant. Therefore, the set of equations (3.27-3.32) contains at most four independent equations for the three quantities of interest.

Now we turn our attention to (3.27,3.28). Before analyzing them for the general case of arbitrary width we briefly discuss their meaning for the quasi-particle regime: In this limit the gradient terms in (3.27) and also the second term on the r.h.s. vanish and this equation becomes a purely algebraic (constraint) equation demanding that $S^<$ ought to be proportional to an on-shell $\delta$-function. In other words this equation reduces to the mass-shell constraint discussed e.g. in [8–11]. The traditional on-shell transport equation is recovered from (3.28) by integrating over the energy [2]. As already pointed out the l.h.s. provides the drift terms. The first two terms on the r.h.s. yield the collision terms as will be discussed in more detail below. The third term on the r.h.s. gives the Vlasov contribution. The last term on the r.h.s. vanishes in the quasi-particle limit. Up to now the role of this term in an off-shell transport theory is not clear. However, once the equations of motion for test particles will have been written down the meaning of this term will become obvious.

Let us now come back to the more general case of arbitrary width. Both equations (3.27,3.28) mix zeroth and first order gradient terms. We first concentrate on the zeroth order terms:

$$\left(p_0 - \frac{\vec{p}^2}{2m} - {\rm Re}\bar{\Sigma}^{\rm ret}\right) S^< = \pm i\bar{\Sigma}^< \,{\rm Re}\bar{D}^{\rm ret} + o(\partial_X)\,, \tag{3.38}$$

$$\Gamma\, S^< = \pm i\bar{\Sigma}^<\, \mathcal{A} + o(\partial_X)\,. \tag{3.39}$$

Since we already know expressions for ${\rm Re}\bar{D}^{\rm ret}$ and $\mathcal{A}$ we insert (3.33,3.34) and obtain

$$\left(p_0 - \frac{\vec{p}^2}{2m} - {\rm Re}\bar{\Sigma}^{\rm ret}\right) S^< = \pm i\bar{\Sigma}^< \,\frac{p_0 - \frac{\vec{p}^2}{2m} - {\rm Re}\bar{\Sigma}^{\rm ret}}{\left(p_0 - \frac{\vec{p}^2}{2m} - {\rm Re}\bar{\Sigma}^{\rm ret}\right)^2 + \frac{1}{4}\Gamma^2} + o(\partial_X)\,, \tag{3.40}$$

$$\Gamma\, S^< = \pm i\bar{\Sigma}^< \,\frac{\Gamma}{\left(p_0 - \frac{\vec{p}^2}{2m} - {\rm Re}\bar{\Sigma}^{\rm ret}\right)^2 + \frac{1}{4}\Gamma^2} + o(\partial_X)\,. \tag{3.41}$$

At first glance these two equations seem to contain the same information in zeroth order gradient expansion:

$$S^< = \frac{\pm i\bar{\Sigma}^<}{\left(p_0 - \frac{\vec{p}^2}{2m} - \mathrm{Re}\bar{\Sigma}^{\mathrm{ret}}\right)^2 + \frac{1}{4}\Gamma^2} + o(\partial_X) \tag{3.42}$$

obtained from (3.40) by dividing by $(p_0 - \frac{\vec{p}^2}{2m} - \mathrm{Re}\bar{\Sigma}^{\mathrm{ret}})$ and from (3.41) by dividing by $\Gamma$. However, we have implicitly assumed in these steps that both factors are not too small, i.e.

$$\frac{1}{p_0 - \frac{\vec{p}^2}{2m} - \mathrm{Re}\bar{\Sigma}^{\mathrm{ret}}}\, o(\partial_X) = o(\partial_X)\,, \tag{3.43}$$

$$\frac{1}{\Gamma}\, o(\partial_X) = o(\partial_X)\,. \tag{3.44}$$

This can only be true if we are *not* in the quasi-particle regime. There, the width is very small. Therefore, e.g. (3.44) might be invalid. Fortunately we are here interested in the case where the width is finite.[1] Thus, (3.42) holds and provides information about the size of a specific combination of Green functions and self-energies:

$$\pm\, i\bar{\Sigma}^< - d\, S^< = o(\partial_X) \tag{3.45}$$

with

$$d = \left(p_0 - \frac{\vec{p}^2}{2m} - \mathrm{Re}\bar{\Sigma}^{\mathrm{ret}}\right)^2 + \frac{1}{4}\Gamma^2\,. \tag{3.46}$$

Note that $d$ is the denominator of $\mathcal{A}$ as well as $\mathrm{Re}\bar{D}^{\mathrm{ret}}$. Since all our consideration are based on the assumption that space-time derivatives are small we conclude that the l.h.s. of (3.45) also has to be small. This is a consistency condition which might be checked in actual calculations. Since we keep terms linear in the gradients the l.h.s. of (3.45) in general cannot be neglected. However, when it appears in gradients we can neglect such a combination, i.e. for arbitrary $B$:

$$[\pm i\bar{\Sigma}^<,\, B] = [d\, S^<,\, B] + \underbrace{[\pm i\bar{\Sigma}^< - d\, S^<,\, B]}_{o(\partial_X^2)} = [d\, S^<,\, B] \tag{3.47}$$

where we have neglected the gradient which is effectively of second order in the gradient expansion [7]. Therefore, it turns out to be consistent with the gradient expansion up to (including) first order to replace $\pm i\bar{\Sigma}^<$ in the first order gradients in (3.27,3.28) by $d\, S^<$. Then, after some straightforward manipulations we find that both equations reduce to the same transport equation (cf. [16])

$$\frac{1}{2}\Gamma\mathcal{A}\,[p_0 - \frac{\vec{p}^2}{2m} - \mathrm{Re}\bar{\Sigma}^{\mathrm{ret}},\, S^<] - \frac{1}{2}\mathcal{A}\,[\Gamma,\, (p_0 - \frac{\vec{p}^2}{2m} - \mathrm{Re}\bar{\Sigma}^{\mathrm{ret}})S^<] = \Gamma\, S^< \mp i\bar{\Sigma}^<\mathcal{A}\,. \tag{3.48}$$

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[1] Nonetheless, in the end we want to have an equation which is also valid for small width, i.e. shows the correct quasi-particle limit. Indeed, as we shall see below, this requirement is fulfilled. Since a transport equation for the quasi-particle limit is well-known we can treat in our derivation the case where the width is large.

With this equation we have reached our goal to derive in first order gradient expansion consistent equations for the three quantities $S^<$, $\mathcal{A}$, and $\mathrm{Re}\bar{D}^{\mathrm{ret}}$. These equations are given by (3.48), (3.33), and (3.34), respectively. They have also been derived in [16,17]. It is interesting to note that only the equation for $S^<$ is a differential equation.

Let us recall again that the derivation of this latter equation has been obtained in a two-step procedure. The first step was the formal gradient expansion yielding *two* equations for $S^<$, namely (3.27) and (3.28). The second step was to realize that a special combination (3.45) of self-energies and Green functions is effectively of first order in the gradient expansion. Without that second step the two equations (3.27,3.28) would not yield identical results and it would be unclear which equation one should use to evolve $S^<$ in time. Of course, it is interesting to find out for which physical situations (3.45) is realized since in turn this indicates where the gradient expansion is justified. For that purpose we rewrite condition (3.45) in the way as it appears in the traditional transport equation (3.28):

$$\Gamma\, S^< \mp i\bar{\Sigma}^<\, \mathcal{A} = \Gamma\, \mathcal{A}\, \left( \frac{S^<}{\mathcal{A}} - \frac{\pm i\bar{\Sigma}^<}{\Gamma} \right) = o(\partial_X)\,. \tag{3.49}$$

Obviously, there are two cases when this special combination of self-energies and two-point functions is small. The first case is the traditional quasi-particle regime. There the width $\Gamma$ is small and from

$$\Gamma = o(\partial_X) \tag{3.50}$$

one might justify the gradient expansion.[2] The second case, relevant for our purpose of describing states with potentially large width, concerns situations where the difference of the fractions in the brackets in (3.49) is small. Actually these two fractions define off-shell generalizations of phase-space densities, one given by the two-point functions

$$n(X,p) = \frac{S^<(X,p)}{\mathcal{A}(X,p)} \tag{3.51}$$

and one demanded by the self-energies

$$n_\Sigma(X,p) = \frac{\pm i\bar{\Sigma}^<(X,p)}{\Gamma(X,p)}\,. \tag{3.52}$$

It is the difference of these two functions which in general determines the time evolution of the system under consideration (see also e.g. [15,11,16]). The smallness of this difference is an alternative possible justification of the gradient expansion if the width of the states is not small. Indeed, this difference is small near thermal equilibrium since there the collisional self-energy is given by [2]

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[2]Naively one might suspect that the spectral function becoming sharply peaked invalidates the given argument in the region of interest, i.e. near the on-shell point. However, in this case one should rather compare energy integrated expressions. Then upon integration over the energy the spectral function in front of the brackets in (3.49) is basically replaced by one.

$$\pm\, i\bar{\Sigma}^<(p) = n_{\rm B,F}(p_0)\,\Gamma(p) \tag{3.53}$$

while $S^<$ is given in (3.17). This shows that both $n$ and $n_\Sigma$ approach the same thermal distribution. The fact that (3.49) is fulfilled near thermal equilibrium does *not* automatically imply that it is not fulfilled outside of this regime, i.e. far away from thermal equilibrium. Whether (3.49) is realized or not for a given physical situation has to be checked in actual calculations. If it was violated the consequence would be that the process at hand cannot be described by first-order gradient expanded equations.

The transport equation (3.48) is the main result of this section. To gain more insight into this equation we first rewrite the r.h.s. in a more common form. Using (3.13,3.26) we get

$$\Gamma\, S^< \mp i\bar{\Sigma}^<\mathcal{A} = i\bar{\Sigma}^> S^< \mp i\bar{\Sigma}^< S^> \tag{3.54}$$

which is the well-known collision term, the difference between loss and gain terms. In the test particle description we are aiming at, the l.h.s. of (3.48) describes the propagation of the test particles. For a first interpretation of these terms on the l.h.s. we study the quasi-particle limit. In spite of the fact that the derivation of (3.48) is doubtful for infinitesimally small width (as outlined above) we will find that (3.48) nonetheless reproduces the correct transport equation for the quasi-particle regime. In this regime one may define an on-shell phase-space density $n$ via

$$S^<(X,p) = n(X; p_0 = \epsilon(X,\vec{p}),\vec{p})\,\mathcal{A}(X,p) \tag{3.55}$$

where $\epsilon$ is the on-shell energy, i.e. the solution $p_0 = \epsilon$ of the equation

$$p_0 - \frac{\vec{p}^2}{2m} - {\rm Re}\bar{\Sigma}^{\rm ret}(X,p) = 0\,. \tag{3.56}$$

In this way (3.48) can be transformed into a transport equation for $n$. We integrate the transport equation over $p_0$ and note that for vanishing width both $\mathcal{A}$ and $\frac{1}{2}\Gamma\mathcal{A}^2$ approach the same $\delta$-function:

$$\left.\begin{matrix}\mathcal{A}\\ \frac{1}{2}\Gamma\mathcal{A}^2\end{matrix}\right\} \to 2\pi\,\delta(p_0 - \frac{\vec{p}^2}{2m} - {\rm Re}\bar{\Sigma}^{\rm ret}) = 2\pi\, z\,\delta(p_0-\epsilon) \quad \text{for} \quad \Gamma\to 0 \tag{3.57}$$

with the renormalization factor [7]

$$z(X,\vec{p}) = \left(1 - \frac{\partial {\rm Re}\bar{\Sigma}^{\rm ret}(X,p)}{\partial p_0}\right)^{-1}\Bigg|_{p_0=\epsilon}\,. \tag{3.58}$$

The collision term, i.e. the r.h.s. of (3.48), becomes after integrating over $p_0$:

$$-\,z\left(\pm i\bar{\Sigma}^<(1\pm n) - i\bar{\Sigma}^> n\right)\Big|_{p_0=\epsilon}\,. \tag{3.59}$$

The first term on the l.h.s. of (3.48) yields the Vlasov part (see also (3.24)):

$$[p_0 - \epsilon,\, n]\Big|_{p_0=\epsilon} . \tag{3.60}$$

Note that the $z$-factor is implicitly incorporated here on account of

$$\vec{\nabla}_X \epsilon = z\vec{\nabla}_X \mathrm{Re}\bar{\Sigma}^{\mathrm{ret}}\Big|_{p_0=\epsilon} \tag{3.61}$$

and corresponding equations for derivatives with respect to momenta. The second term on the l.h.s. of (3.48) is a genuine off-shell contribution. After performing the integration over $p_0$ this term vanishes for $\Gamma \to 0$. The origin of this term can be traced back to the last term on the r.h.s. of (3.28). As already emphasized the role of this term in an off-shell transport theory still requires a proper interpretation which we will give below in the test particle picture. To summarize, we have recovered from (3.48) the traditional Boltzmann equation in the quasi-particle limit.

In the next sections we will elaborate on the more general case of non-vanishing width. Among other things we will address two questions: 1. Is the particle number still a conserved quantity when the transport equation (3.48) is used to describe the time evolution of $S^<$? 2. What is the role of the off-shell contribution in (3.48)?

## IV. EFFECTIVE PARTICLE NUMBER

We are aiming at a test particle representation for the transport process described by (3.48). For a test particle representation to make sense it is mandatory that the represented quantity is conserved as outlined in Sec. II. Suppose that we have chosen the self-energies such (conserving approximation [2,7,16]) that the particle number (3.16) is conserved for the full quantum theory defined by the exact equations of motion (3.1,3.2). This does not necessarily mean that also the transport equation (3.48) conserves this quantity. Indeed, as we will argue below this is not the case. The reason is simply that by the coarse graining process carried out by the gradient expansion the part of $S^<(X,p)$ which is highly oscillating in $X$ has been neglected. Of course, the temporal deviation in the particle number is effectively of higher order in the gradient expansion since up to first order the exact and the approximate equations of motion carry the same information. However, it would be very unpleasant if one had to represent a quantity by test particles which is not fully conserved on the level of approximations. Thus, after showing that the particle number is not fully conserved we shall search for another quantity which is exactly conserved by the transport equation.

To study the time evolution of the particle number (3.16) we divide (3.48) by $\frac{1}{2}\Gamma\mathcal{A}$ and integrate over the space coordinates and the four-momentum. All gradients with respect to $\vec{x}$ and $\vec{p}$ vanish by partial integration (and neglecting surface terms). Finally we get

$$\begin{aligned}\frac{d}{dt}N(t) &= \frac{d}{dt}\int d^3x \int \frac{d^4p}{(2\pi)^4} S^<(t,\vec{x};p) \\ &= \frac{d}{dt}\int d^3x \int \frac{d^4p}{(2\pi)^4} S^< K + \int d^3x \int \frac{d^4p}{(2\pi)^4}\frac{2}{\Gamma\mathcal{A}}\left(\pm i\bar{\Sigma}^< S^> - i\bar{\Sigma}^> S^<\right)\end{aligned} \tag{4.1}$$

with

$$K(X,p) = \frac{\partial \mathrm{Re}\bar{\Sigma}^{\mathrm{ret}}(X,p)}{\partial p_0} + \frac{p_0 - \frac{\vec{p}^2}{2m} - \mathrm{Re}\bar{\Sigma}^{\mathrm{ret}}(X,p)}{\Gamma(X,p)} \frac{\partial \Gamma(X,p)}{\partial p_0} \,. \tag{4.2}$$

Note that the quantity $1-K$ is the off-shell generalization of the inverse quasi-particle renormalization factor $1/z$ as defined in (3.58).

Now we have to specify the self-energies. To keep things simple we choose a two-body interaction

$$\frac{1}{2}\int d^3x\, d^3x'\, \psi^\dagger(t,\vec{x})\, \psi^\dagger(t,\vec{x}')\, v(|\vec{x}-\vec{x}'|)\, \psi(t,\vec{x}')\, \psi(t,\vec{x}) \tag{4.3}$$

and evaluate the collisional self-energies in the Born approximation [2,5]

$$\begin{aligned} i\bar{\Sigma}^{>}(X,p) = & \int \frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}\frac{d^4p_3}{(2\pi)^4}\,(2\pi)^4\delta^{(4)}(p+p_1-p_2-p_3) \\ & \times \frac{1}{2}\left(\bar{v}(\vec{p}-\vec{p}_2) \pm \bar{v}(\vec{p}-\vec{p}_3)\right)^2 S^{<}(X,p_1)\, S^{>}(X,p_2)\, S^{>}(X,p_3)\,, \end{aligned} \tag{4.4}$$

$$\begin{aligned} \pm i\bar{\Sigma}^{<}(X,p) = & \int \frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}\frac{d^4p_3}{(2\pi)^4}\,(2\pi)^4\delta^{(4)}(p+p_1-p_2-p_3) \\ & \times \frac{1}{2}\left(\bar{v}(\vec{p}-\vec{p}_2) \pm \bar{v}(\vec{p}-\vec{p}_3)\right)^2 S^{>}(X,p_1)\, S^{<}(X,p_2)\, S^{<}(X,p_3)\,. \end{aligned} \tag{4.5}$$

For a more elaborate treatment of self-energies we refer to [16].

The two contributions on the r.h.s. of (4.1) show completely different structures. Especially we note that the first term belongs to the part of the transport equation which describes the propagation of modes while the second term comes from the collisional part. It is hard to conceive that these two terms can cancel each other. Indeed, we have not managed to rearrange these terms in a way that one can get an idea why the r.h.s. of (4.1) should vanish. We therefore strongly conjecture that $N(t)$ is *not* conserved by the transport equation (3.48). At this stage a comparison to the approach of [7] is in order. There terms like the first one on the r.h.s. of (4.1) are neglected since it contains a double derivative — one with respect to $t$ and one with respect to $p_0$. We cannot neglect such contributions since we have only assumed that space-time derivatives are small. We have not made any assumptions about the smallness of derivatives with respect to momenta. Even more important, we are not aiming at a quantity which is nearly conserved by the transport equation up to terms which are of higher order in the gradients. To represent the transport equation by test particles we need a quantity which is exactly conserved. Of course, this difference is somewhat subtle since e.g. the particle number is exactly conserved by the full quantum mechanical equation of motion. However, it is not the latter equation we want to solve by a test particle ansatz but the transport equation obtained by gradient expansion.

We will now construct a quantity which is conserved by (3.48). The first step is to get rid of the collisional part. By inspecting (4.4,4.5) we observe that the following integral vanishes [2,7]:

$$\int \frac{d^4p}{(2\pi)^4}\left(\pm i\bar{\Sigma}^{<}S^{>} - i\bar{\Sigma}^{>}S^{<}\right) = 0\,. \tag{4.6}$$

This suggests to directly integrate (3.48) over momenta and space coordinates instead of dividing it by $\frac{1}{2}\Gamma\mathcal{A}$ first. Indeed, for the quantity

$$\mathcal{S} = \frac{1}{2}\Gamma\mathcal{A}S^< \tag{4.7}$$

we derive from (3.48) the following equation of motion (see Appendix B):

$$[p_0 - \frac{\vec{p}^2}{2m} - \mathrm{Re}\bar{\Sigma}^{\mathrm{ret}},\, \mathcal{S}] - \frac{1}{\Gamma}\,[\Gamma,\, (p_0 - \frac{\vec{p}^2}{2m} - \mathrm{Re}\bar{\Sigma}^{\mathrm{ret}})\mathcal{S}] = i\bar{\Sigma}^> S^< \mp i\bar{\Sigma}^< S^> \,. \tag{4.8}$$

Thus we find

$$\frac{d}{dt}\int d^3x \int \frac{d^4p}{(2\pi)^4}\mathcal{S} = \frac{d}{dt}\int d^3x \int \frac{d^4p}{(2\pi)^4}\mathcal{S}K \,. \tag{4.9}$$

We conclude that the space and momentum integral of the quantity

$$\tilde{S}^< = \mathcal{S}\,(1-K) = \frac{1}{2}\Gamma\mathcal{A}S^<\,(1-K) \tag{4.10}$$

is conserved by the transport equation (3.48):

$$\frac{d}{dt}\tilde{N}(t) = \frac{d}{dt}\int d^3x \int \frac{d^4p}{(2\pi)^4}\tilde{S}^<(t,\vec{x};p) = \int d^3x \int \frac{d^4p}{(2\pi)^4}\left(\pm i\bar{\Sigma}^< S^> - i\bar{\Sigma}^> S^<\right) = 0\,. \tag{4.11}$$

In the following we will refer to $\tilde{N}$ as the effective particle number. In the quasi-particle limit, $\Gamma \to 0$, we find

$$\tilde{S}^<(X,p) \to n(X;\epsilon,\vec{p})\,2\pi\,\delta(p_0-\epsilon) \tag{4.12}$$

where we have used (3.55,3.57) and

$$1 - K \to z^{-1}\,. \tag{4.13}$$

Thus, in the quasi-particle limit there remains an interesting difference between

$$S^<(X,p) \to n(X;\epsilon,\vec{p})\,2\pi\,z\,\delta(p_0-\epsilon) \tag{4.14}$$

and $\tilde{S}^<$. The quantity which "counts" the quasi-particles is indeed $\tilde{S}^<$ while the renormalization factor $z$ appears in $S^<$ which takes into account that there is some strength in the spectral function (far) away from the quasi-particle pole.

As a second limiting case we evaluate $\tilde{S}^<$ for thermal equilibrium. Using (3.17) we get

$$\tilde{S}^<_{\mathrm{th}}(p) = n_{\mathrm{B,F}}(p_0)\,\frac{1}{2}\Gamma(p)\mathcal{A}^2(p)\,(1-K(p))\,. \tag{4.15}$$

For constant width and vanishing $\mathrm{Re}\bar{\Sigma}^{\mathrm{ret}}$ the quantities $S^<_{\mathrm{th}}$ and $\tilde{S}^<_{\mathrm{th}}$ are depicted in Fig. 1. Obviously the function $\frac{1}{2}\Gamma\mathcal{A}^2$ is more strongly peaked than $\mathcal{A}$ (also cf. [12]). The asymmetric forms are caused by the Bose enhancement.

Comparison of (4.15) with (3.17) suggests to introduce — also for non-equilibrium situations — an effective spectral function

$$\tilde{\mathcal{A}}(X,p) = \frac{1}{2}\Gamma(X,p)\,\mathcal{A}^2(X,p)\,(1-K(X,p))\,. \tag{4.16}$$

Indeed, as will be shown in Appendix A the effective spectral function is also normalized (cf. (3.15)). Thus we have a correspondence of the quantities $S^<$, $N$, and $\mathcal{A}$ of the full theory with the quantities $\tilde{S}^<$, $\tilde{N}$, and $\tilde{\mathcal{A}}$ of the coarse grained theory obtained by gradient expansion. However, a word of caution is in order here. For the interpretation of $\tilde{\mathcal{A}}$ as an effective spectral function (and also of $\tilde{N}$ as an effective particle number!) it is mandatory that $\tilde{\mathcal{A}}$ is always non-negative. This may not be the case for arbitrary self-energies. Since $\Gamma$ is always positive one has to make sure in actual calculations that $1-K$ is non-negative. Otherwise a test particle ansatz would not make any sense. Finally we note that $\tilde{\mathcal{A}}$ was also introduced in [16] as the spectral information which enters the entropy density.

## V. EQUATIONS OF MOTION FOR TEST PARTICLES

The next step is the test particle ansatz for $\tilde{S}^<$ and the derivation of the equations of motion for the test particles from the transport equation (3.48):

$$\tilde{S}^<(t,\vec{x};p) \sim \sum_i \delta^{(3)}(\vec{x}-\vec{x}_i(t))\,\delta(p_0-E_i(t))\,\delta^{(3)}(\vec{p}-\vec{p}_i(t))\,. \tag{5.1}$$

Note that the energy $E_i$ of the test particle $i$ is a free coordinate, not restricted by a mass-shell condition. Instead of (3.48) we use the equivalent equation (4.8). While the r.h.s. describes the collisions the l.h.s. will give the propagation of the test particles between collisions. To get the equation of motion for a single test particle *between two collisions* we have to insert

$$\mathcal{S}(t,\vec{x};p) = \frac{\tilde{S}^<(t,\vec{x};p)}{1-K(t,\vec{x};p)} \sim \sum_i \frac{1}{1-K(t,\vec{x}_i;E_i,\vec{p}_i)}\delta^{(3)}(\vec{x}-\vec{x}_i(t))\,\delta(p_0-E_i(t))\,\delta^{(3)}(\vec{p}-\vec{p}_i(t)) \tag{5.2}$$

into

$$[p_0 - \frac{\vec{p}^2}{2m} - \mathrm{Re}\bar{\Sigma}^{\mathrm{ret}},\,\mathcal{S}] - \frac{1}{\Gamma}\,[\Gamma,\,(p_0 - \frac{\vec{p}^2}{2m} - \mathrm{Re}\bar{\Sigma}^{\mathrm{ret}})\mathcal{S}] = 0\,. \tag{5.3}$$

In general this yields an equation of the following type

$$\sum_i \Big(a(t,\vec{x}_i;E_i,\vec{p}_i) + b(t,\vec{x}_i;E_i,\vec{p}_i)\partial_{p_0} + \vec{c}(t,\vec{x}_i;E_i,\vec{p}_i)\vec{\nabla}_x + \vec{d}(t,\vec{x}_i;E_i,\vec{p}_i)\vec{\nabla}_p\Big) \otimes \delta^{(3)}(\vec{x}-\vec{x}_i)\,\delta(p_0-E_i)\,\delta^{(3)}(\vec{p}-\vec{p}_i) = 0\,. \tag{5.4}$$

A solution for this equation is obtained by demanding that all coefficient functions $a$, $b$, $\vec{c}$, and $\vec{d}$ have to vanish. In general this gives eight equations for the seven test particle coordinates $E_i$, $\vec{x}_i$, and $\vec{p}_i$. If and only if the quantity (in our case $\tilde{S}^<$) which is represented

by test particles corresponds to a conserved quantity (here $\tilde{N}$) then the coefficient $a$ in (5.4) vanishes and one only has to fulfill seven equations instead of eight. This is the reason why we have insisted to find an exactly conserved quantity. Otherwise one would have to deal with an overdetermined system of equations. Neglecting one of the obtained equations would not yield a solution for the transport equation one actually wants to solve (cf. the corresponding discussion in Sec. II). For the case at hand we get the following equations of motion for the test particles

$$0 \stackrel{!}{=} -\dot{E}_i + \frac{1}{1-K}\left(\partial_t \mathrm{Re}\bar{\Sigma}^{\mathrm{ret}} + \frac{E_i - \frac{\vec{p}_i^2}{2m} - \mathrm{Re}\bar{\Sigma}^{\mathrm{ret}}}{\Gamma}\partial_t\Gamma\right) = b(t,\vec{x}_i;E_i,\vec{p}_i)\,, \tag{5.5}$$

$$0 \stackrel{!}{=} -\dot{\vec{x}}_i + \frac{1}{1-K}\left(\frac{\vec{p}_i}{m} + \vec{\nabla}_{p_i}\mathrm{Re}\bar{\Sigma}^{\mathrm{ret}} + \frac{E_i - \frac{\vec{p}_i^2}{2m} - \mathrm{Re}\bar{\Sigma}^{\mathrm{ret}}}{\Gamma}\vec{\nabla}_{p_i}\Gamma\right) = \vec{c}(t,\vec{x}_i;E_i,\vec{p}_i)\,, \tag{5.6}$$

$$0 \stackrel{!}{=} -\dot{\vec{p}}_i - \frac{1}{1-K}\left(\vec{\nabla}_{x_i}\mathrm{Re}\bar{\Sigma}^{\mathrm{ret}} + \frac{E_i - \frac{\vec{p}_i^2}{2m} - \mathrm{Re}\bar{\Sigma}^{\mathrm{ret}}}{\Gamma}\vec{\nabla}_{x_i}\Gamma\right) = \vec{d}(t,\vec{x}_i;E_i,\vec{p}_i)\,. \tag{5.7}$$

Newton's equations of motion are obtained by neglecting $K$, i.e. putting the renormalization factor $1/(1-K)$ to 1, and disregarding all terms where the width $\Gamma$ enters. In this limiting case one observes that the energy of a test particle is only changed if the real part of the self-energy, i.e. the classical potential, depends explicitly on time. If the potential shows an explicit momentum dependence the classical expression for the velocity is modified accordingly. For finite width but neglecting the energy and momentum dependence of the self-energies (5.5-5.7) basically reduce to the test particle equations presented in [17]. The only remaining difference is that there a relativistic system is studied.

Let us now come back to the full equations of motion (5.5-5.7). Obviously the terms which involve the width yield genuine off-shell contributions to all equations of motion since on the mass-shell the combination $E_i - \frac{\vec{p}_i^2}{2m} - \mathrm{Re}\bar{\Sigma}^{\mathrm{ret}}$ vanishes by definition (cf. (3.56)). It is interesting to calculate the time evolution of the off-shellness of a test particle defined by

$$\Delta E_i(t,\vec{x}_i;E_i,\vec{p}_i) = E_i - \frac{\vec{p}_i^2}{2m} - \mathrm{Re}\bar{\Sigma}^{\mathrm{ret}}(t,\vec{x}_i;E_i,\vec{p}_i)\,. \tag{5.8}$$

From the equations of motion (5.5-5.7) we find

$$\frac{d}{dt}\Delta E_i = \frac{\Delta E_i}{\Gamma}\frac{d}{dt}\Gamma\,. \tag{5.9}$$

This equation has also been presented in [17] and in [18]. To the best of our knowledge the full set of equations of motion (5.5-5.7) has never been derived before. Obviously the time evolution of the off-shellness is caused by the respective last term in the brackets on the l.h.s. of (5.5-5.7). These contributions can be traced back to the second term on the l.h.s. of (3.48). This clarifies the meaning of the latter term for an off-shell transport theory: It provides the time evolution of the off-shellness. Actually this off-shell contribution in the transport equation (3.48) is caused by the respective last term on the r.h.s. of (3.27) and (3.28) via the replacement (3.47). Without that latter replacement the meaning of these terms in (3.27,3.28) would have been completely unclear.

The equations of motion (5.5-5.7) obeyed by the test particles between collisions are the main result of this section. It is useful to show that they reproduce the correct quasi-particle limit. The quasi-particle energy of a test particle is defined as the solution $\epsilon_i = \epsilon_i(t, \vec{x}_i; \vec{p}_i)$ of the equation

$$\epsilon_i - \frac{\vec{p}_i^{\,2}}{2m} - \mathrm{Re}\bar{\Sigma}^{\mathrm{ret}}(t, \vec{x}_i; \epsilon_i, \vec{p}_i) = 0\,. \tag{5.10}$$

Differentiating (5.10) e.g. with respect to $\vec{p}_i$ yields (also cf. (3.61))

$$\vec{\nabla}_{p_i}\epsilon_i = \frac{1}{1 - K(t, \vec{x}_i; \epsilon_i, \vec{p}_i)} \left( \frac{\vec{p}_i}{m} + \vec{\nabla}_{p_i}\mathrm{Re}\bar{\Sigma}^{\mathrm{ret}}(t, \vec{x}_i; \epsilon_i, \vec{p}_i) \right) . \tag{5.11}$$

Thus we find the expected relations

$$\dot{\vec{x}}_i = \vec{\nabla}_{p_i}\epsilon_i\,, \tag{5.12}$$

$$\dot{\vec{p}}_i = -\vec{\nabla}_{x_i}\epsilon_i\,. \tag{5.13}$$

The time evolution of the on-shell energy is governed by

$$\dot{\epsilon}_i = \frac{1}{1 - K(t, \vec{x}_i; \epsilon_i, \vec{p}_i)} \partial_t \mathrm{Re}\bar{\Sigma}^{\mathrm{ret}}(t, \vec{x}_i; \epsilon_i, \vec{p}_i) = z(t, \vec{x}_i; \epsilon_i, \vec{p}_i)\, \partial_t \mathrm{Re}\bar{\Sigma}^{\mathrm{ret}}(t, \vec{x}_i; \epsilon_i, \vec{p}_i)\,. \tag{5.14}$$

Finally we comment on the evaluation of the self-energies which enter the equations of motion (5.5-5.7) and of course also the collisional part on the r.h.s. of the transport equation (3.48). Except for the Hartree-Fock self-energy which is comparatively easy to incorporate in transport calculations all self-energy contributions can be traced back to the determination of $\pm i\bar{\Sigma}^<$ and $i\bar{\Sigma}^>$ via the relations (3.26,3.36,4.2). In the Born approximation to two-body collisions these self-energies are given in (4.4,4.5). These quantities should be expressed in terms of $\tilde{S}^<$ since the latter quantity is represented by test particles according to (5.1). We find for the Born rates

$$\begin{aligned} i\bar{\Sigma}^>(X,p) = &\int \frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}\frac{d^4p_3}{(2\pi)^4}\,(2\pi)^4\delta^{(4)}(p + p_1 - p_2 - p_3)\,\frac{1}{2}\,(\bar{v}(\vec{p} - \vec{p}_2) \pm \bar{v}(\vec{p} - \vec{p}_3))^2 \\ &\times \frac{\tilde{S}_1^<}{\frac{1}{2}\Gamma_1\,\mathcal{A}_1\,(1 - K_1)}\,\frac{\tilde{\mathcal{A}}_2 \pm \tilde{S}_2^<}{\frac{1}{2}\Gamma_2\,\mathcal{A}_2\,(1 - K_2)}\,\frac{\tilde{\mathcal{A}}_3 \pm \tilde{S}_3^<}{\frac{1}{2}\Gamma_3\,\mathcal{A}_3\,(1 - K_3)}\,, \end{aligned} \tag{5.15}$$

$$\begin{aligned} \pm i\bar{\Sigma}^<(X,p) = &\int \frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}\frac{d^4p_3}{(2\pi)^4}\,(2\pi)^4\delta^{(4)}(p + p_1 - p_2 - p_3)\,\frac{1}{2}\,(\bar{v}(\vec{p} - \vec{p}_2) \pm \bar{v}(\vec{p} - \vec{p}_3))^2 \\ &\times \frac{\tilde{\mathcal{A}}_1 \pm \tilde{S}_1^<}{\frac{1}{2}\Gamma_1\,\mathcal{A}_1\,(1 - K_1)}\,\frac{\tilde{S}_2^<}{\frac{1}{2}\Gamma_2\,\mathcal{A}_2\,(1 - K_2)}\,\frac{\tilde{S}_3^<}{\frac{1}{2}\Gamma_3\,\mathcal{A}_3\,(1 - K_3)} \end{aligned} \tag{5.16}$$

with $\tilde{S}_j^< = \tilde{S}^<(X, p_j)$ etc. Obviously the fact that $\tilde{S}^<$ instead of $S^<$ is represented by test particles causes rather non-trivial modifications for the collision integrals. In turn, disregarding these modifications amounts to solving transport equations which might be rather different from the ones one actually wants to solve.

## VI. SUMMARY AND OUTLOOK

In this work we have presented a derivation of a transport equation making use of the gradient expansion but without utilizing the commonly used quasi-particle approximation. Therefore, the derived equations can be used for the description of processes where the width of the involved modes might be arbitrarily large.[3] In principle, the gradient expansion yields six equations for the three quantities of interest. We have successively shown which of these equations are redundant by various rearrangements. In the course of this procedure it has become apparent that a specific combination of self-energies and Green functions — which is seemingly of zeroth order in the gradient expansion — must be treated as being of first order. This identification (3.45) is mandatory to derive a contribution to the transport equation which finally leads to the description of the time evolution of the off-shellness of the test particles. As a next step a quantity — the effective particle number — has been isolated which is exactly conserved by the transport equation. This has opened the way to a test particle representation of the density $\tilde{S}^<$ which corresponds to the effective particle number. The equations of motion for the test particles have been presented. Finally the Born collision rates are evaluated within the test particle representation.

Remarkably the effective particle number is *not* identical to the particle number which is exactly conserved by the fully quantum field theoretical equations of motion. Of course, the particle number is approximately conserved by the transport equation since up to first order in the gradient expansion the approximate (transport) and the exact (quantum field theoretical) equation carry the same information. To apply a test particle ansatz to the transport equation one needs, however, a quantity which is exactly conserved. Nonetheless, in actual simulations the particle number $N$ of the full theory might serve as a test for the accuracy of the gradient expansion. If during the time evolution $N$ deviates drastically from its initial value the gradient expansion has to be regarded as inappropriate for the system one wants to simulate.

Throughout this work we have strictly distinguished between the (possibly off-shell) propagation of the test particles between collisions on the one hand side and the binary collisions on the other. This distinction might seem to be artificial since the width which allows for an off-shell propagation is caused by the collisions. Since our equations are derived from the underlying quantum field theory in a well-defined and controlled way we do not have to be afraid of double counting. The physical picture we have in mind which corresponds to our approach are collisions which are so frequent that particles do not come back to their mass-shell before the next collision happens. Therefore part of the collision process has to be incorporated in the propagation by allowing the propagation of test particles with arbitrary energy. An additional aspect which we have not touched here but is in principle straightforward to include is the treatment of resonances with large decay width. Here it is very natural that a propagating resonance is simulated by a bunch of test particles with variable energies. We note that our formalism treats collisional and decay width on equal

[3]Of course, for a non-relativistic formalism to make sense one has to assume that kinetic and potential energy and also the width is small as compared to the rest mass of the particle.

footing.

We would like to comment briefly on the possible numerical realization of the presented approach. To calculate for a given space-time point the spectral function and the width which enter the collision integrals (5.15,5.16) one has to know the self-energies for arbitrary four-momenta. This requires the use of much more test particles as compared to simulations which are restricted to the quasi-particle regime. In present simulations of off-shell effects in transport theories rather ad-hoc recipes are in use to overcome this problem. In [17] the self-energies are not determined independently for every point in space but rather from an average over a larger volume. Whether this is a valid approximation to the full treatment of the self-energies remains to be seen. In [18] the self-energies for a given space-time point are parametrized by an expression adopted from the case of thermal equilibrium. The thermal parameters are locally chosen such that a reasonable fit on the shape suggested by the test particles is obtained. It has been checked there that time as well as momentum averages of the parametrized self-energies agree with the corresponding results from the test particle representation.

Next we compare our formalism to recent other approaches. In [16] it has been worked out in great detail how general self-energies have to be chosen such that particle number and energy are exactly conserved by the full quantum field theoretical Kadanoff-Baym equations. It has not been shown there that these quantities are *exactly* conserved by the approximate gradient expanded equations. As we have outlined above we conjecture that this is indeed not the case. To some extent it is trivial that quantities which are exactly conserved by the exact equations are (at least) approximately conserved by the approximate equations. Nonetheless it is interesting to see how this approximate conservation comes about in the approximate transport formalism [7]. In our approach, however, we had to answer a somewhat different question. For the test particle realization we had to find a quantity which is exactly conserved by the approximate transport equation.

The purpose of [17] and [18] was also to present a test particle description of off-shell transport processes. In spite of the fact that in [17] the transport equation (3.48) was derived in basically the same way as presented here, the approach of [17] is very different in spirit. There the gradient expanded Kadanoff-Baym equations were only used to determine the evolution of test particles between collisions. In contrast, the information about how collisions between these test particles have to be treated was taken from a completely different source. Since in [17] it was anyway not the purpose to solve equation (3.48) it did not matter whether the test particle equations of motion were derived from a test particle ansatz for $S^<$ or for $\tilde{S}^<$. Obviously, in the approach presented here we take the transport equation (3.48) serious as being an equation which can be obtained from the underlying quantum field theoretical equations in a straightforward and well-controlled manner by gradient expansion. A second difference between our approach and the one presented in [17] is the fact that the energy-momentum dependence of the self-energies was neglected for the test particle equations of motion. Therefore, the test particle equations presented here are more general than the ones given in [17].[4] Finally we studied here a non-relativistic system

[4]After finishing this work a preprint [21] appeared where the authors of [17] generalized their

while in [17] the case of relativistic bosons was treated.

In [18] the starting point was the transport equation (3.28) but neglecting the last term on the r.h.s. The equation (5.9) which describes the time evolution of the off-shellness was introduced by hand based on physical plausibility arguments. The collision terms were treated in the way adopted from on-shell transport theories by simulating $S^<$ instead of $\tilde{S}^<$ by test particles.

While discussing at length the (non-)conservation of the particle number we have not touched the issue of energy conservation. In view of the fact that it was non-trivial to find a conserved effective particle number it might be no surprise to realize that also the conservation of energy as defined for the full quantum field theoretical equations becomes a problem for the gradient expanded equations. To clarify the important question of energy conservation is beyond the scope of the present paper. We only want to note here that the energy as defined for the full theory (see e.g. [2,7,16]) has to be at least approximately conserved by the transport equation. In principle this provides a check on the validity of the gradient expansion.

Finally we give a brief outlook on the generalization of our approach to relativistic systems. Most of the presented formalism can be immediately generalized, especially for scalar particles. As already mentioned above (after Eq. (3.37)) it might appear for fermions that the set of four equations for spectral function and real part of the retarded propagator derived in first order gradient expansion cannot be trivially reduced to two independent equations. This might lead to new constraints for self-energies and Green functions. A second potential problem of a relativistic off-shell transport theory concerns the possibility of space-like modes. In principle, the spectral function might contain strength in the sector where the momentum is larger than the energy. Thus it can happen that in collisions space-like test particles are created which travel faster than the speed of light. This is clearly an unpleasant feature of a theory which should respect causality. For relativistic systems one therefore has to think about the proper treatment of the space-like part of the spectral function.

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relativistic test particle equations by also including energy-momentum dependences of the self-energies.

## APPENDIX A: NORMALIZATION OF SPECTRAL FUNCTION AND EFFECTIVE SPECTRAL FUNCTION

In this appendix we will prove the normalization of the spectral function $\mathcal{A}$ as given in (3.33) and the effective spectral function $\tilde{\mathcal{A}}$ as defined in (4.16). We implicitly assume in the following that $\tilde{\mathcal{A}}$ is always non-negative (cf. the remarks after (4.16)). For the spectral function we find

$$\int_{-\infty}^{\infty} \frac{dp_0}{2\pi} \mathcal{A}(X,p) = -2\mathrm{Im} \int_{-\infty}^{\infty} \frac{dp_0}{2\pi} \bar{D}^{\mathrm{ret}}(X,p)\,. \tag{A1}$$

Since $\bar{D}^{\mathrm{ret}}(X,p)$ has only poles in the lower complex half-plane one can use contour integration to evaluate the last integral. We assume that the self-energies can be neglected for large $|p_0|$, i.e.

$$\bar{D}^{\mathrm{ret}}(X,p) \to \frac{1}{p_0} \qquad \text{for} \quad |p_0| \to \infty\,. \tag{A2}$$

This allows us to write

$$\int_{-\infty}^{\infty} \frac{dp_0}{2\pi} \mathcal{A}(X,p) = -2\mathrm{Im} \left( \oint_{\mathcal{C}_1+\mathcal{C}_2} \frac{dp_0}{2\pi} \bar{D}^{\mathrm{ret}}(X,p) - \int_{\mathcal{C}_2} \frac{dp_0}{2\pi} \frac{1}{p_0} \right) = -2\mathrm{Im} \left( 0 - \frac{1}{2\pi} \int_0^{\pi} d\phi\, i \right) = 1 \tag{A3}$$

where $\mathcal{C}_1$ is a path along the real axis and $\mathcal{C}_2$ is an infinitely large half-circle surrounding counter clockwise the upper complex half-plane.

For the effective spectral function we note that $1-K$ can be written as (cf. (4.2))

$$1 - K = \Gamma \frac{\partial r}{\partial p_0} \tag{A4}$$

with the ratio

$$r(X,p) = \frac{p_0 - \frac{\vec{p}^2}{2m} - \mathrm{Re}\bar{\Sigma}^{\mathrm{ret}}(X,p)}{\Gamma(X,p)}\,. \tag{A5}$$

Thus we get

$$\tilde{\mathcal{A}} = \frac{1}{2}\Gamma \mathcal{A}^2 (1-K) = \frac{1}{2} \frac{1}{(r^2+\frac{1}{4})^2} \frac{\partial r}{\partial p_0}\,. \tag{A6}$$

Now the normalization of the effective spectral function can easily be shown:

$$\int_{-\infty}^{\infty} \frac{dp_0}{2\pi} \tilde{\mathcal{A}} = \int_{-\infty}^{\infty} \frac{dr}{2\pi} \frac{1}{2} \frac{1}{(r^2+\frac{1}{4})^2} = 1 \tag{A7}$$

where we have used that

$$r \to \pm\infty \qquad \text{for} \qquad p_0 \to \pm\infty\,. \tag{A8}$$

Again we have assumed that for large $|p_0|$ the self-energies can be neglected as compared to $p_0$.

## APPENDIX B: EQUATION OF MOTION FOR $\mathcal{S}$

In this Appendix we will derive (4.8) from (3.48). We have to show that

$$\begin{aligned}
&[p_0 - \frac{\vec{p}^2}{2m} - \mathrm{Re}\bar{\Sigma}^{\mathrm{ret}},\, \mathcal{S}] - \frac{1}{\Gamma}\,[\Gamma,\, (p_0 - \frac{\vec{p}^2}{2m} - \mathrm{Re}\bar{\Sigma}^{\mathrm{ret}})\mathcal{S}] \\
&- \frac{1}{2}\Gamma\mathcal{A}\,[p_0 - \frac{\vec{p}^2}{2m} - \mathrm{Re}\bar{\Sigma}^{\mathrm{ret}},\, S^<] + \frac{1}{2}\mathcal{A}\,[\Gamma,\, (p_0 - \frac{\vec{p}^2}{2m} - \mathrm{Re}\bar{\Sigma}^{\mathrm{ret}})S^<]
\end{aligned} \tag{B1}$$

vanishes. Using the definition (4.7) we find that (B1) reduces to

$$\begin{aligned}
&[p_0 - \frac{\vec{p}^2}{2m} - \mathrm{Re}\bar{\Sigma}^{\mathrm{ret}},\, \frac{1}{2}\Gamma\mathcal{A}]\, S^< - \frac{p_0 - \frac{\vec{p}^2}{2m} - \mathrm{Re}\bar{\Sigma}^{\mathrm{ret}}}{\Gamma}\,[\Gamma,\, \frac{1}{2}\Gamma\mathcal{A}]\, S^< \\
&= (p_0 - \frac{\vec{p}^2}{2m} - \mathrm{Re}\bar{\Sigma}^{\mathrm{ret}})\, S^<\, [\log\frac{p_0 - \frac{\vec{p}^2}{2m} - \mathrm{Re}\bar{\Sigma}^{\mathrm{ret}}}{\Gamma},\, \frac{1}{2}\Gamma\mathcal{A}] \\
&= (p_0 - \frac{\vec{p}^2}{2m} - \mathrm{Re}\bar{\Sigma}^{\mathrm{ret}})\, S^<\, [\log(r),\, \frac{1}{2}\frac{1}{r^2 + \frac{1}{4}}]
\end{aligned} \tag{B2}$$

with the ratio $r$ defined in (A5). To finish our proof we only have to note that the last Poisson bracket in (B2) vanishes on account of the following general property of the Poisson bracket:

$$[f_1(g(X,p)),\, f_2(g(X,p))] = 0 \tag{B3}$$

for arbitrary functions $g$ and arbitrary functions $f_1$, $f_2$ which do not explicitly depend on $X$ and $p$ but only implicitly via $g$.

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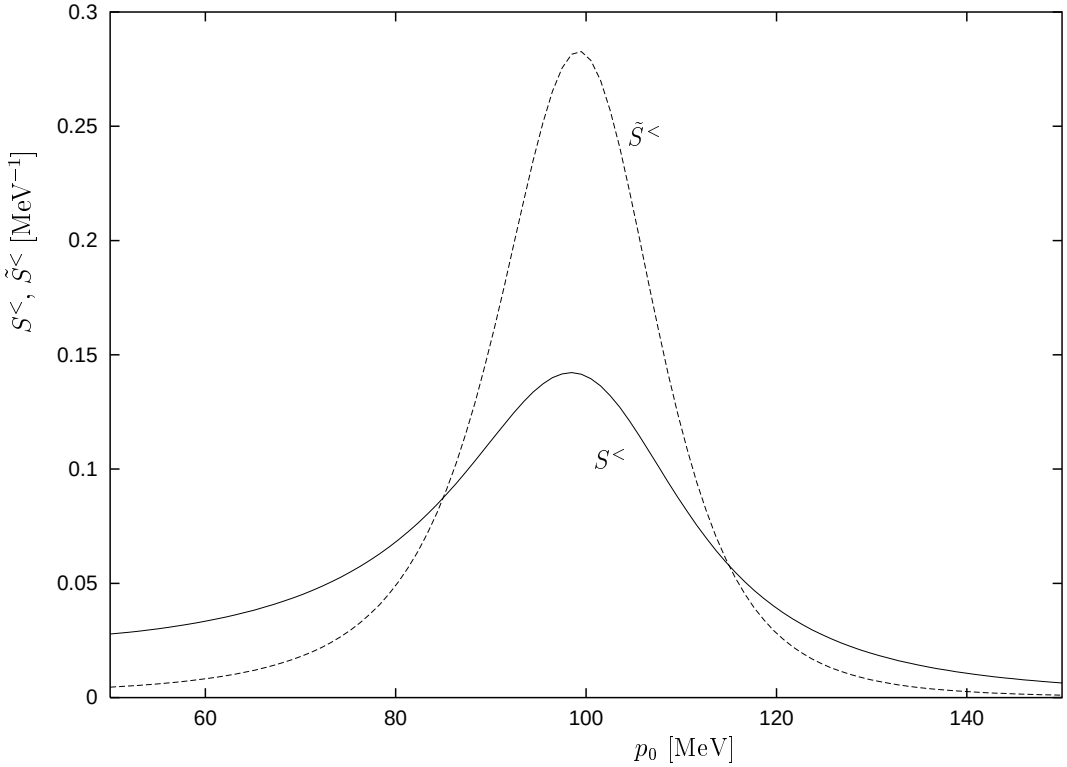


FIG. 1. $S^<_{\rm th}$ (full) and $\tilde{S}^<_{\rm th}$ (dashed) as functions of the energy $p_0$ for the case of bosons. The width is chosen to be constant, $\Gamma = 30\,\mathrm{MeV}$. The temperature is $T = 150\,\mathrm{MeV}$ and the kinetic energy $\frac{\vec{p}^2}{2m} = 100\,\mathrm{MeV}$. See main text for more details.